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Investigating transgressive actions undertaken by students while studying mathematics via primary sources

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ABSTRACT

We describe an exploration informed by transgressionism and supported by data collected from students' report of learning abstract algebra via curriculum materials where primary historical sources are central to their design, namely Primary Source Projects (PSPs). PSPs require students to actively engage with primary source texts written by historical mathematicians and counter students' more traditional mathematical learning experiences. We explore the components forming transgressive triads consisting of *outcomes* of *transgressive actions* taken to overcome *boundaries* encountered in students' study of mathematics. Particularly, we focus on transgressive actions undertaken by students and provide exemplars identified from data to empirically support our theoretical conceptualisations. We end with a discussion by proposing research that draws upon transgressions in both learning mathematics via PHS in particular and with regard to other approaches in undergraduate mathematics education which may place students in contexts in which their familiar is made unfamiliar.

Keywords: transgressions, primary historical sources, undergraduate mathematics education

INTRODUCTION

Mathematicians, mathematics education researchers, and mathematics historians have advocated for the use of history of mathematics in teaching and learning mathematics for decades (Clark et al., 2019; Fauvel, 1991; Jankvist, 2009). Since 2000, there has been an increase in research concerning the domain of history in mathematics education (see Clark et al., 2018 for a survey of recent developments). For example, one project focused on creating and site testing primary source materials for use in the undergraduate mathematics curriculum. As part of that effort, researchers have examined different aspects of how studying mathematical concepts via primary source materials contributes to students' learning experiences (Clark et al., 2022).

Research literature from a variety of perspectives has documented the challenges students face when learning mathematics via primary historical sources (PHS). For example, anecdotally, Lappa and Nikolantonakis (2018) described how the supportive nature of a collaborative teaching method enabled the historical texts to "overcome the difficulties that arose" and function "as cognitive tools in the course of students' conceptual understanding" (p. 339). Jahnke et al. (2000) theorised regarding the impact on student learning via different perspectives on the effects of using primary sources in teaching by conceptualising these perspectives as *replacement*, *reorientation*, and *cultural understanding*. Replacement is characterised by replacing the usual with something different; reorientation is characterised as challenging perceptions "by making the familiar unfamiliar" (Jahnke et al., 2022, p. 1431; e.g., seeing mathematics as "strings of words" instead of algebraic symbols and numbers), and finally, cultural understanding as associated by placing "the development of mathematics in a scientific, technological, or societal context of a given time and place and in the history of ideas and society" (Jankvist, 2014, p. 880). However, such perspectives are not gained without challenges. Rather trivially, "the presence of an unknown language is a quite typical obstacle" (Jahnke et al., 2000, p. 312). As well, interpreting a primary source requires the reader to "engage with a complex network of relations between their own interpretations of a certain concept… and the interpretations of the original author" (Jahnke et al., 2000, p. 298). Glaubitz (2010), in his description of such challenges to using primary sources in teaching, noted that:

> … the study of original sources requires teachers and students to be prepared to dive into some strange and unknown realm of thinking, to appreciate cultural and historical context and–last but not least–to deal competently with written text that is more extensive than the word problems they are used to in mathematics (p. 352).

Glaubitz (2010) characterised his own teaching experiment, which involved a unit that made use of a historical genetic approach, as "a complete failure" (p. 355); however, other investigations have offered more positive results. Many recent efforts have been directed towards students' learning of metadiscursive rules (or, metarules) based on the participationist framework on learning utilising Sfard's (2008) theory of thinking as communication. For example, several studies have focused on the potential for primary sources to promote students' reflections on metarules in mathematics (e.g., Kjeldsen & Petersen, 2014), to "diagnose participants' metarules in teaching situations where historical sources are investigated" (Bernardes & Roque, 2018, p. 225), and to explore how primary sources can support students' "figuring out" of metadiscursive rules (Barnett et al., 2021).

Therefore, we acknowledge the nebulous yet well-established existence of challenges surrounding learning mathematics via PHS, and in this article we propose how the theoretical lens of *transgressions* (Kozielecki, 1986) is useful for unpacking the ways in which students overcome related challenges to reach otherwise inaccessible outcomes. The goal of this article is to focus our attention on transgressive actions which students undertake during their study of undergraduate mathematics via PHS. We build on previous work focused on transgressions in the learning of mathematics by asserting that transgressions lead to discernible outcomes which may be grounded in situated learning theory due to the nature of the data with which we punctuate our claims. To do so, we offer several exemplars from existing data. The data supporting our exploration were collected as part of the TRIUMPHS grant project, which began in 2015 and was an eight-year[1], seven-institution effort funded by the United States National Science Foundation. The project's primary goal was to develop, field test, and disseminate materials based upon PHS for teaching standard topics in university mathematics courses. These materials, called Primary Source Projects (PSPs), contain excerpts from historical sources, discussion of the mathematical significance and relevant historical context, and student tasks designed to illuminate the mathematical concepts of focus in the sources.

In the design of PSPs, the central idea is to challenge students to be active interpreters of the texts as part of their learning of modern mathematics via a "read, reflect, respond" (Barnett et al., 2014, p. 10) approach to the sources. Despite differences in PSPs due to, for example, intended course or PSP author expertise, students' reading of the original source excerpts are interrupted via frequent tasks to help students "develop their own understanding of concepts and theory, and to reflect from the primary source on the evolution of standards and rigor, and of the nature of proof, definitions, notation, terminology" (Barnett et al., 2016, p. 10).

As part of the development of their own understanding, reading of original sources enables students to assume the role of thinking as others (i.e., historical figures in mathematics):

> This thinking into other persons and into a different world seems to be the core of an educational philosophy underlying the reading of original sources. She who thinks herself into a scientist doing mathematics at a different time has herself to do mathematics; she moves in a mental game in the primary circle reflecting what the person under study might have had in mind. One has to ask for the theoretical conditions this person is explicitly or implicitly supposing, and one will have to mobilise imagination to generate hypotheses about them (Jahnke et al., 2000, p. 299).

As well, students participating as active interpreters and thinking as others are prominent practices associated with the implementation of PSPs and substantially contribute to initiatives "for undergraduate mathematics to move away from a traditional lecture model" (Melhuish et al., 2023, p. 1) that still persists at university. This is particularly relevant when considering students' early experiences in university mathematics courses presents them with learning environments characterised by increased lecture and decreased opportunities for active learning. Thus, students' study of mathematics using PSPs has significant potential for students engaging in these particular mathematical practices not afforded by more traditional instructional models.

In the next section, we outline the central constructs of the theory based on a psychological model conceptualised by Kozielecki (1986) that guided our empirical analysis. To account for changes in students' experiences due to a central aspect of the model, we use situated learning theory to identify learning outcomes as new having/being for its relevance in both teaching with PSPs and the notion of having/being envisioned in the proposed framework.

THEORETICAL BACKGROUND

Transgressions Framework

The term *transgression* is often considered with a negative connotation, such as violating the norms of a standard practice (especially in the extreme example of committing a crime or sin), but in its most neutral form, transgression simply refers to the action of moving beyond a limit or boundary. Most notably, Kozielecki (1986), developed a psychological model of expansive human behaviour, *transgressionism*, in which he dichotomized actions to be either protective or transgressive in nature. The former are actions which aim to "preserve the status quo" (p. 90). The latter term, and focus of our paper, includes purposeful actions that lead "to an outcome exceeding the boundaries of the individual's past achievements" (p. 89). To Kozielecki (1986), such boundaries were a "demarcation line that marks out the scope and the type of positive or negative value scored by the person so far" (p. 90). In this sense, transgressing could be a positive or negative action. That is, going against the norm has the potential to be constructive or destructive. Thus, in the context of learning mathematics in an "abnormal" setting, we find value in considering how students' actions in such settings can be constructive.

As stated by Clark et al. (2022), Kozielecki's (1986) notion of transgressing against a norm (i.e., violating a norm to exceed previously established boundaries) is well-suited to exploring how students attain outcomes resulting from navigating unfamiliar

[1] Student data were collected only during the first five years of the project (2015-2020).

learning experiences such as studying mathematics via PSPs. We envision that with similar non-traditional instructional methods, students engage in actions that may either exceed barriers, which may have been in place due to years of studying mathematics in certain ways (drilling practice problems, listening to lecture, etc.), or students may engage in (protective) actions in which they try to maintain their existing state of how they learn mathematics. We have seen evidence of both types of actions in our own teaching experience as well as in data collected from implementation of PSPs in undergraduate mathematics classrooms. We propose that examining the process of moving beyond barriers which are tied to learning via non-traditional methods of mathematics instruction can be useful in the study of mathematics learning as well as informing future mathematics instruction which employ less traditional instructional methods.

Regarding Kozielecki's (1986) notion of transgressionism, in which the focus is on the journey of the individual, we highlight three components which are at the core of these highly individualised experiences: *boundaries*, *transgressive* (or *protective*) *actions*[2], and *outcomes*, which we refer to as a transgressive triad. We theorise that by considering these three components together, researchers may be afforded a panoramic perspective of mathematics learning, especially mathematics learning that includes non-traditional instruction.

Considering transgressions in a mathematics education context is not entirely novel; however, our proposal of examining mathematics learning via what we term as *transgressive triads* builds on the work of mathematics education researchers who have explored transgressions, albeit in a broader sense. Semadeni (2015) introduced the notion of *mathematical cognitive transgressions* in which an individual crosses a "previously non-traversable limit of own mathematical knowledge or of a previous barrier of deep-rooted convictions" (p. 27). In his investigation, Semadeni (2015) theorised instances of mathematical transgressions in areas of calculus, arithmetic, and geometry. For example, Leibniz transgressed to a new state of being by associating areas under curves with tangents to curves, while mathematicians like Pascal were bound by their own mathematical knowledge and did not transgress. Even more broad was Semadeni's (2015) consideration that a "significant transgression" (p. 32) occurred when the philosophical community began to develop general theorems about ideal geometric objects instead of reasoning solely about imperfect observable geometric objects. This phenomenon was notably discussed by Poincaré (1902/1905), who proposed that experiments with observable solids were and are productive in reasoning about the ideal geometrical space. For someone who, "as yet, know[s] nothing of geometry" and "whose ideas of space are not yet formed" (p. 60), beginning with experimental facts may reveal some aspects of the ideal geometric space, leading them to discover otherwise inaccessible properties of the geometry. The transgression, therefore, exists and must lie somewhere in the transition from observable to non-observable geometry.

Similarly, Lakoma (2016) theorised the existence of transgressive actions leading from a person's previous state of understanding a "finite probability space to [an] infinite probability space" (p. 470). She referenced Huygens' (1657) work as being the first documented example of solving a problem using an infinite probability space. Common among these general examples presented by Semadeni (2015) and Lakoma (2016) is that transgressions often brought a whole community to a new state of being, e.g., the community of philosophers, scientists, or the scientific community as a whole. The researchers also juxtaposed the previous state (what was surpassed) with the present state, what we consider, respectively, as the boundary and the outcome to provide theoretical evidence for transgression. Additionally, transgressive actions were never specified, yet we suggest that transgressive actions, albeit on a smaller scale, may be undertaken by individuals of a community and are thus associated with comparatively finer boundaries and outcomes.

Recently, transgressive actions were studied at a grain size finer than that described above. Pieronkiewicz (2015a) restricted her focus to *affective transgressions*, which she defined as the "intentional process of overcoming personal affective barriers that preclude one's mathematical growth and development" (p. 1263). Although this notion of transgressions is centred around the realm of affect, Pieronkiewicz (2015a) was the first researcher we found to propose perceptible actions embodied as transgressions to overcome barriers. These barriers are associated with affect as well and include negative feelings regarding one's mathematical ability, e.g., self-stigmatisation. The transgressive action itself comprised both a "corrective emotional experience and meta-affect" (Pieronkiewicz, 2015b, Affective transgression in learning mathematics, para. 5). Additionally, what motivates a person to transgress to a positive affective state has been theorised as partly originating from fundamental needs, desires, and goals (Goldin, 2020). This *conative transgression* is an "experience of desires and/or embrace of goals… that were previously 'off limits' to the person in some important way" (Goldin, 2020, p. 69). In the context of learning mathematics via PSPs, students' fulfilment of such basic desires may be achieved by experiencing the "beauty, insight, power, and connection" (p. 65) of mathematics when studied through the lens of a notable historical figure. While we recognise that consideration of students' motivation to engage in transgressive actions while studying mathematics via PSPs can inform aspects of the transgressive triad, exploring motivation in this regard is beyond the intent of this paper.

Notably, the transgressive action does not always originate with the student's own needs and desires. Pieronkiewicz and Goldin (2020) asserted that "the immediate incentive to challenge the status quo may originate with the external facilitator" (p. 77), or to borrow Wachtel's (1991) term, the *accomplice*, i.e., the teacher acting as the accomplice may intervene and instigate a change. We support the stance that the accomplice can initiate a transgressive action in learning mathematics, and we broaden the reach of an accomplice, capable of both "preventing and facilitating the change" (Pieronkiewicz, 2015b, Affective transgression in learning mathematics, para. 3), to include teachers, instructional tools, and curriculum materials, such as PSPs.

Accomplices as external influences activating a transgressive action in learning mathematics have been recently empirically supported. For instance, Pieronkiewicz and Goldin (2020) presented the case of Sarah who experienced affective mathematical

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[2] Although we do not have the space to investigate protective actions, we acknowledge that they may similarly exist in a triad connecting barriers and outcomes.

transgressions because of a persistent tutor who "encouraged Sarah to face her [mathematically affective] problems and to see that she herself was capable of solving them" (p. 82). The tutor in this case was the accomplice for affective transgressions. Further evidence provided by Watford (2022) supports that people can act as the accomplice to mathematical transgressions more broadly. University students in a history of calculus course undertook transgressive actions by engaging in modes of discourse with the instructor and peers in small groups to overcome boundaries such as unfamiliar language, confidence, and anxiety (Watford, 2022). As well, the accomplice may not always be a person. Duda (2020) reported that an instructional tool played a similar role in that a mere graphing calculator was the "stimulator of students' creative transgressions" (p. 267) when solving non-routine algebra problems. Therefore, we propose that if people and instructional tools can promote mathematical transgressions, then curriculum materials, such as PSPs, may do so as well.

Our proposed transgressive triad model offers a new perspective to studying transgressive actions, on a similar level. We recognise that the examples above present different grain sizes which can be studied via transgressions, and we theorise that the finer the grain-size, the more detailed and well-defined the triad. Although we propose the existence of varying degrees of transgressive action (and thus triad) grain size, our intention is not to focus on determining appropriate grain size for study. Rather, we draw attention to previous research (e.g., Lakoma, 2016; Semadeni, 2015) that reported the existence of much more global transgressive actions and conjecture that any associated boundaries and outcomes would also be at a similar global scale. Thus, by studying and empirically identifying transgressive actions and their associated boundaries and outcomes comprising triads at a more local scale, researchers may be better informed about how individuals (students) overcome barriers to mathematics learning and what outcomes from doing so are possible. Below, we describe the components of a transgressive triad in more detail, and later, we offer an example of a triad founded in data from the TRIUMPHS project.

The Transgressive Triad

Researchers agree that transgressive actions are purposeful actions; however, transgressive actions may still be undertaken without complete awareness of the scope of the boundary or any possible outcomes. There are varying levels of consciousness when engaging in transgressive actions (Semadeni, 2015). For example, children who transgress from being able to reason only with concrete numbers (number words associated with physical objects being counted), to being able to reason with abstract numbers (number words not associated with any physical quantity), are unconscious of their transgression (Semadeni, 2015). They are able to appreciate their new state of being but are unaware of the previous state of being of not reasoning with abstract numbers. Likewise, students may be semi-conscious of their transgressive action in which they "could be aware of it" (p. 27) but do not think about it at the time of transgression. Therefore, students may be fully conscious, semi-conscious, or unconscious of their transgressive actions. We suggest that many transgressive actions regarding learning mathematics via less traditional instructional methods, like PSPs, may be semi-conscious, and students become aware of the action, outcome, and (possibly) boundary when prompted through some investigative medium such as a survey or interview.

To transgress, an individual must traverse a boundary that leads to some outcome which exceeds their past achievements. Although Semadeni's (2015) notion of a boundary as a limit to one's "own mathematical knowledge" or "deep rooted convictions" (p. 27) is rather broad, it allows the identification of any factor which may contribute to students' current mathematics potential. We conjecture there are two types of boundaries associated with mathematics learning: internal and external. Internal boundaries are closely related to affective components such as thoughts, feelings, and emotions but may also be concerned with students' perception of mathematics and what they expect mathematics learning to look like. External boundaries are related to aspects of mathematics that students have not been exposed to, such as mathematical content, or in the case of PSPs, ways of reading and doing mathematics. Due to the nature of boundaries being connected with students' experiences in learning mathematics, discerning a particular boundary may be difficult. Semadeni (2015) wrote, "what gets surpassed … is always somehow integrated with the newly constructed structure" (p. 28), indicating the close relationship between barriers and outcomes. Thus, describing a boundary in relation to the transgressive action and outcome via triads is useful and follows naturally from previous research which generally characterised a *before* and *after* the transgressive action was undertaken.

Outcomes have been conceptualised as a person's new "having or being" (Kozielecki, 1986, p. 89) which result from crossing the boundaries of their past achievements. In essence, the outcome is the "after" of the transgressive action. Furthermore, Semadeni (2015) claimed that the barrier crossing must be from a "specific lower level to a new specific upper level" (p. 27). This notion was expanded by Lakoma (2016) to a state in which students are able to "think and reason at the more advanced and mature level" (p. 466). Considering examples discussed above, outcomes may be of the form of reasoning in the ideal geometric space, as with Poincaré (1902/1905), or Sarah's (Pieronkiewicz & Goldin, 2020) new affective state in which she believes that she is capable of doing mathematics. We support the notion that the outcome must be some discernible new (to the student) level and propose that due to the relationship between boundaries and outcomes as described above, the outcome, when linked in a triad, is expected to be closely related to the barrier. An example triad signifying this relationship is explored in the *Featuring "Participation" in a Transgressive Triad* section.

Situated Theory of Learning

Kozielecki (1986) theorised the nature of outcomes is "more complex and less well-defined" (p. 102), a phenomenon we experienced as we struggled to tease apart and thematise outcomes from the data. In this sense, we propose that another framework, situated learning theory, provides a useful lens to capture distinct outcomes as overarching themes. Using constructs such as access and participation, we provide a foundation for our claims regarding students' traversing boundaries related to their mathematical experiences.

Essential to the definition of *learning* in situated theory is learners' *access* to activities of a particular community of practice. Having access "to a wide range of ongoing activity, old-timers, and other members of the community; and to information,

Table 1. Post-PSP implementation: Open-ended student survey items

Open ended survey items
(OE1) Please comment on how your understanding of abstract algebra in particular or mathematics in general has changed as a result of this PSP.
(OE2) Please comment on how your work on this PSP has changed your attitudes toward mathematics in general or towards the study of abstract algebra in particular.
(OE3) In your opinion, what are the benefits of learning mathematics by reading the historical sources in this PSP?
(OE4) In your opinion, what are the drawbacks of learning mathematics by reading the historical sources in this PSP?
(OE5) Please describe the experience you have had this semester (as a student) with using historical materials (e.g., primary source materials) in an undergraduate mathematics course.
(OE6) One of your friends is planning to take a mathematics course that will use a PSP (or projects). What would you say to your friend about what they could expect to do and to learn from this experience?

resources, and opportunities for participation" (Lave & Wenger, 1991, p. 101), newcomers "gradually assemble a general idea of what constitutes the practice of the community" (p. 95). Through their increasing participation, newcomers (i.e., learners) become more knowledgeable about what it means to be a member of the related community of practice (Lave, 2019).

Participation is key to learning as it is also the process by which students develop their practice (Handley et al., 2007). *Practice* in the context of a community of practice goes beyond what an instructor presents in the classroom and includes everyday activities of old-timers. Activities include "how, when, and about what [they] collaborate, collude, and collide, and what they enjoy, dislike, respect, and admire" (Lave & Wenger, 1991, p. 95). With respect to mathematical activity, Hersh (1991) distinguished the "front" of mathematics from the "back" of mathematics:

> The 'front' of mathematics is mathematics in 'finished' form, as it is presented to the public in classrooms, textbooks, and journals. The 'back' would be mathematics as it appears among working mathematicians, in informal settings, told to another in an office behind closed doors (p. 128).

This distinction (i.e., front and back) limits outsiders' ability to develop a realistic understanding of mathematics and leads to generalisations such as "mathematical truth or knowledge is the same for everyone. It does not depend on who in particular discovers it; in fact, it is true whether or not anybody ever discovers it" (Hersh, 1991, p. 130). Hersh (1991) argued that to develop an expert-like perspective about mathematics, one needs to "move from the 'front' to the 'back'" (p. 132).

For newcomers to become full participants of a particular community of practice, it is essential that these learners have access to not only a general body of activities of old-timers in that mathematical community, but also opportunities to participate in related activities. Subsequently, learners will cultivate an understanding of the activities of the practitioners and develop their practice. Thus, learners are situated within a community of practice and engage in activities with those more expert than themselves. In our research, those experts also include historical mathematicians in the form of excerpts from their writings.

METHODOLOGY

Classroom Context and Participants

The data we drew upon for this article were collected in 2018 from four abstract algebra classrooms. These classrooms were of particular interest to the TRIUMPHS project team because of the need to collect student data on two PSPs (Barnett, 2018a, 2018b), which were new to the TRIUMPHS collection at the time. The four instructors (Drs. Elwood[3], Frost, Guillen, and Hunter) agreed to participate as site testers, and subsequently, we collected data from 54 consenting students (approximately equally distributed across the four course sites), 33% of whom were pure mathematics majors. Students pursuing an undergraduate degree in pure mathematics typically complete an abstract algebra course during the third or fourth year of a 4-year program.

Data Sources

Several data sources informed several empirical investigations during the first five years of the TRIUMPHS project; however, the primary data which served as the core of conceptualising about the transgressive actions which students undertake while studying mathematics via PSPs are the six open-ended items (**Table 1**) which appeared on student surveys (implemented after completing each PSP and at the conclusion of the course) and were designed to allow students to report on their learning abstract algebra specifically, mathematics more generally, and using PHS. We also collected implementation reports from site testing instructors, detailing, for example, number of class sessions, additions or omissions of tasks or readings within an individual PSP, as well as pertinent information regarding an instructor's typical teaching practice and what changed as they implemented a particular PSP (see for example, the summary of implementation features provided in **Table 2**).

Implementation of PSPs

Extensive commentary on the instructional practices of the instructors participating in the TRIUMPHS project is beyond the scope of what we seek to accomplish in the present article. However, without exception in the contexts from which our supporting empirical evidence was collected, each instructor's practice was driven by their desire for students to discuss and make sense of (both individually and in shared classroom spaces) the texts they were studying. Furthermore, each instructor established

[3] All names used in this paper are pseudonyms.

Table 2. Features of four instructors' PSP implementation

Implementation feature	Instructor
Timing of PSP: Used to introduce topics in the course	Dr. Elwood, Dr. Frost, & Dr. Guillen
Timing of PSP: Used to review topics in the course	Dr. Hunter
Provided additional historical context or content	Dr. Frost & Dr. Guillen
Facilitated whole-class reading of PSP	Dr. Guillen
Assigned "advanced work" readings and/or tasks to complete before coming to class	Dr. Elwood, Dr. Frost, Dr. Guillen, & Dr. Hunter
Utilised individual thinking/small-group work/whole-class discussion approach for PSP tasks	Dr. Elwood, Dr. Frost, & Dr. Guillen
Worked on PSP tasks as a class	Dr. Hunter

classrooms which enabled shared readings, question posing, interpretation, and discussion of the mathematical ideas that were at the core of the PSPs implemented. It was clear from the instructors' perspective that they sought to avoid what Sfard (2014) described as the case of:

> … even if familiar with almost every word that comes out of the lecturer's mouth, new students often feel that the words fail to make much sense when put next to one another. To many novices, those lectures are a traumatic experience … (p. 199).

As the TRIUMPHS project has reported elsewhere (Can, 2019; Mauntel, 2018), there are certainly differences in how PSPs are implemented in university mathematics classrooms. Such differences may be due to the PSP materials themselves, instructor goals, instructor familiarity with and interest in history of mathematics, or some combination of these factors. We emphasise that one of our aims was to propose a theoretical lens for the special actions students choose to undertake while learning university mathematics via PSPs; however, we provide a brief summary of instructional features common across the four classrooms from which our empirical data were collected (**Table 2**).

Data Analysis

Although this article is designed to help focus on aspects of the transgressions framework as they relate to students' learning experiences while studying undergraduate mathematics topics via PSPs, we note that data were not collected with the transgressions framework in mind, and thus, we recognise the limitation that this presents. For example, the open-ended survey items were written with a much more global goal in mind (i.e., investigating what the study of mathematics via PHS contributed to, or presented obstacles for, students' learning experience in undergraduate mathematics). Consequently, our ability to identify triad components was limited to analysing student responses not originally created for this purpose.

We began with open coding 400 student responses to the open-ended survey items to determine whether we could identify transgressive actions within students' responses. In this first phase of coding, we also identified potential boundaries whenever possible, whereas identification of learning outcomes was not a priority until the final stages of analysis. After this first pass through the data, we formalised the coding strategy for subsequent rounds. This strategy entailed coding student responses for the purpose of identifying transgressive actions and corresponding boundaries. The resulting product was a preliminary codebook of particular transgressive actions undertaken by students which may lead them to (or ends with) crossing a boundary. We note that in early phases of coding that the identification of boundaries was difficult, and in this way, we used the transgressions framework to ground our efforts to identify the boundary that marked the difference between students' past having or being and new having or being. However, in this early work, it became clear that identifying full triads (boundary–transgressive action–outcome) would be challenging, particularly since the original intent of the data collection did not include research situated within the transgressions framework. Instead, this further prompted us to refocus our efforts toward theorisation regarding one key element of the triad–transgressive actions–and toward this end, we passed through our data one additional time to locate exemplars to use in support of the claims discussed in Results section.

RESULTS

Innovation of the Transgressive Lens: Focus on an Illustrative Transgressive Triad

To demonstrate our conception of a transgressive triad (comprising a boundary, transgressive action, and an outcome) we offer an illustrative example of how Ryan, a student in Dr. Fermat's abstract algebra course, which was taught entirely with PSPs, was able to gain access to the particular practice of mathematics as having insight to mathematical thought. Although this outcome is a consequence of crossing over a "previously impassable bound of somebody's mathematical knowledge" (Semadeni, 2015, p. 39), such boundaries are not easily discernible, particularly due to data not capturing students' previous state with regard to the particular outcome. More discernible in the data, however, are outcomes and transgressive actions which we chose to interpret with a lens of situated learning theory to highlight the versatility of the transgressions framework and the nature of mathematics learning via PSPs. For this reason, we begin by presenting Ryan's responses in the next section and discuss the outcome and associated transgressive action for the triad. Then, we conjecture boundaries that Ryan crossed. We conclude our results by offering evidence on a more global level that other students in the abstract algebra courses have transgressed by studying mathematics via PSPs through a matching process based on attributes of transgressions from Pieronkiewicz (2015a, 2015b).

Featuring "Participation" in a Transgressive Triad

We present Ryan's responses to the second survey item at both mid-semester and post-semester, respectively:

> I like it more than ever before; using PSPs gives me a lot of insight into what the mathematicians were thinking and how they went about creating something completely new. When going through each 'task' I get this small sense of accomplishment by solving proofs and making conjectures, giving a sense [of] discovery and fulfilment.
>
> It allows me to think of these mathematicians as more of problem solvers. For example, Dedekind hit a wall when it came to indecomposables, however he sought out to make something completely new that explains where the missing solutions are and restores a great mathematical property.

Based on Ryan's responses, we have identified an outcome and a transgressive action grounded in situated learning theory: *access to practitioner practice* and *participation in the practice of mathematicians*, respectively. We begin the presentation of this illustrative example with a discussion of the outcome and transgressive action pertaining to this transgressive triad. Then, we outline our process for identification of boundaries in general (considering the theory as well as our data) and the boundary to complete this specific triad.

Access as an outcome

The outcome, *access to practitioner practice*, is evidenced as a new level in which students have "insight" into expert mathematical thought, particularly the process of mathematical innovation. Ryan mentioned not only gaining access to expert thinking but also understanding the "how" of mathematical creation, noting the shift in his conception of the work of mathematicians such as Dedekind. This is a striking example of students being exposed to and thereby gaining access to Hersh's (1991) "back" of mathematics "as it appears among working mathematicians" (p. 128). It is therefore no surprise that a student who newly accesses the "back" of mathematics would also display positive affective sensations that Pieronkiewicz (2015a) characterised as being present in an affective transgression. As exemplified above, Ryan indicated he "[liked either mathematics or abstract algebra] more than ever before" and noted particularly the senses of accomplishment, discovery, and fulfilment he experienced while working with the PSP. Such affective experiences may position students closer to insiders within the mathematical community, thereby further providing access to mathematical practice based on their work with the PSP.

Participation in the practice of mathematicians as a transgressive action

Ryan's responses indicate that the tasks of the PSP or the PSP itself was the driving factor to propel him to the outcome of access, and he perceived his engagement with the PSP to occur at a level more deeply than simply responding to tasks. In this sense, the PSPs acted as the accomplice to the transgressive action. That is, the PSPs stimulated Ryan to transgress by engaging in mathematical practices (e.g., proving, discovering, making conjectures, etc.). We argue that Ryan's interaction with the PSPs still constitutes a transgression, albeit in a "semi-conscious" state. Such a transgression is characterised by students being aware of the action occurring but may not fully appreciate how it extended them from a previous to a new state of being (Semadeni, 2015). Although Ryan demonstrated awareness of the transformation, it could be that the survey item prompted Ryan and other students to realise or appreciate the transformation of states. For example, Ryan stated how engaging with PSPs was the impetus for reaching the outcome of *access*. He was much more specific in his description of how the tasks of the PSPs prompted him to participate in the mathematical practices of "solving proofs and making conjectures." It was the aforementioned aspects of participation that led to the outcome. Based on Ryan's transgression through *participation*, which helped him reach the outcome of *access*, we wondered what may have hindered him from experiencing this outcome before studying mathematics via PSPs.

Preface to boundary identification

We began our process of boundary identification with the question: *What hindered or prevented Ryan's learning of mathematics?* In theory, some boundaries are based on affective components, such as confidence and motivation, and what students expect and perceive, such as how mathematics should be learned. Other boundaries are rooted in experiences not easily altered by students and possibly needing the influence of an accomplice like PSPs. It is in this second category that we conjecture Ryan's boundaries reside with respect to the outcome of access.

The boundary discussed below exhibits how PSPs served as a vehicle of passage across boundaries otherwise impassable and potentially imperceptible until encountered. Some boundaries may not be apparent to students until they engage in an experience to overcome a boundary, relevant to Semadeni's (2015) treatment of transgressive actions occurring at varying levels of consciousness. Therefore, it was difficult to find clear, distinguishable components of the triad in our data; so, we had to glean what was holding Ryan back based on the more clearly identified outcome and transgressive action. Thus, we present an informed speculation of the boundary.

Boundary identification: Limited exposure to the activities of expert mathematicians

To better understand what may have hindered Ryan's *access to practitioner practice*, we turn to Semadeni's (2015) notion that "what gets surpassed … is always somehow integrated with the newly constructed structure" (p. 28). We interpret the boundary as having been "surpassed" by the student to the new structure of *access*. In this regard, the outcome may reveal possible boundaries that Ryan encountered when those boundaries were not explicitly stated. Thus, for the highlighted transgressive triad, we identified the boundary *limited exposure to the activities of expert mathematicians* consisting of previous states in which students, for example, did not think of "mathematicians as more of problem solvers" and were not able to see their work first-

Table 3. Protective actions with exemplars (Kozielecki, 1986; Pieronkiewicz, 2015a)

Protective actions	Protective actions exemplars
Play key role in adaptation and survival	I would tell them to take a different class because it would do more harm than good [Elmer, OE6].
Often predictable Undertaken to maintain the status-quo	I really feel I need a more straightforward treatment along with the historical sources. I would like a presentation organized in a way that would serve as a convenient reference. Topic–associated definitions, theorems, formulas, and examples; next topic–associated definitions, theorems, formulas, and examples, etc. (I like little boxes in my textbooks!). Since I am new to the material, I do not have enough perspective to organize the material very well on my own [Stephanie, OE4].
Accompanied by negative emotions, especially fear	I feel an unusually low level of confidence in my grasp of the material. This makes me wonder if I have what it takes to continue in math. I think I would prefer geometry proofs where I can at least draw pictures of what I am trying to prove [Stephanie, OE2].

Note. OE#: The open-ended item number (see **Table 1**)

hand to gain insight into their thinking. In this sense, students' perception of mathematics may resemble a ready-made product (possibly encountered in the form of a textbook) which conceals genuine human activity of struggle and persistence that led to the finished form presented to the student. Ryan's response is not clear about a previous state; however, we can assert that a previous state did exist and did not include the "insight" into mathematical thought and mathematical creation which was gained by transgressing via participating in mathematical practices. The senses of discovery, fulfilment, and accomplishment are also apparently so novel to Ryan that he felt the need to write about such a transition. Therefore, we conjecture that if Ryan had more exposure to the practices of expert mathematicians, he would not have transgressed to reach a "new specific upper level" (Semadeni, 2015, p. 27) of access to practitioner practice.

PSPs Promoting Transgressions

While Ryan's transgressive triad, which features participation as a transgressive action, is just one illustrative example providing evidence that such triads exist and can be captured via the transgressive actions framework, we have also found strong evidence of the transgressive actions component of the triad in the survey responses of other students. Therefore, we continue to punctuate our theory of transgressions students undertake while studying with PSPs by matching aspects of conceptualisation of transgressive versus protective actions (Pieronkiewicz, 2015a, p. 1262) with student survey responses. In **Table 3** and **Table 4**, we only include actions for which we found evidence in the survey data, further suggesting the existence of transgressive actions when studying via PSPs.

Remarkably, some students, such as Alton, exhibited more than one attribute of transgressing. Because he noted feeling excitement and anticipation in studying mathematics, his learning experience with PSPs was perceived as doing mathematics for the sake of a higher enjoyment rather than meeting the minimum requirements of the course. Additionally, Alton reported that studying with PSPs has led to a meaningful change in the clarity of his mathematical communication. This is in stark contrast to Elmer and Spencer who indicated a strong desire for more familiar teaching methods, which would thereby maintain the status quo. Although they did not undertake actions to maintain instructional methods to which they have grown accustomed, their responses suggest they would undertake protective actions, such as avoiding a course taught with PSPs, given the choice.

In addition to students displaying more than one attribute in their responses, some attributes from Pieronkiewicz's (2015a) conceptualisations appeared to overlap, such as "I know I am able to" and "possible." It appears rather trivial that students who know they are able to do something perceive the possibility. However, is the converse true? Do students who perceive possibility know that *they* are the ones who can perform the action? For this reason, we call for a refinement of the ideas offered in **Table 3** and **Table 4**. Our work has built on the ideas in **Table 3** and **Table 4**, but more examples from data aligned with the transgressions framework are needed to expand and operationalise these terms. There is also the potential to add other attributes of transgressive actions.

DISCUSSION

Initially, we wanted to investigate how using PSPs in learning mathematics provides an opportunity for students to transgress in mathematics, thus offering one such example of the explanatory power of transgressions in undergraduate mathematics learning spaces. We have taken our cue from previous theoretical work in transgressions in mathematics education research more broadly (Pieronkiewicz, 2015b) and with regard to mathematical cognitive transgressions and considering the historical development of particular mathematical ideas. In this paper, we introduced the notion of transgressing mathematics learning with the aid of an accomplice in the form of PSPs. We also proposed to examine mathematics learning, especially when that occurs with non-traditional instructional methods, in a more structured transgressive lens through a triad comprising boundary, transgressive action, and outcome. Based on our theoretical exploration grounded in data, we suggest that transgressive actions are undertaken by students who study mathematics via PSPs, and therefore, the associated boundaries and outcomes also exist and warrant further exploration to identify difficulties students encounter and to empirically determine what is possible as a result from studying mathematics via PSPs.

Our decision to explore how transgressions might contribute to mathematics education research initially provided challenges. First, no empirical work existed–in mathematics education more broadly–to serve as examples of research informed by this theoretical lens. Second, we attempted to develop theory based on data originally gathered for another purpose. This limitation actually helped illuminate possibilities for future research. Because of our theoretical venture, we were able to explore what was

Table 4. Transgressive actions with exemplars (Kozielecki, 1986; Pieronkiewicz, 2015a)

Transgressive actions	Transgressive actions exemplars
Satisfy higher needs of a human being	Very enjoyable. I liked the different aspect of learning a new mathematical concept. It is almost always expected that we have: lecture, reading, examples, and then homework, in that order. To have it instead be a mix of all four elements was informative and exciting. I looked forward to math because of how engaging it was, something I hadn't felt in quite some time [Alton, OE5].
Harder to predict	While reading the historical sources, I would sometimes get lost and confused in the language used to describe the steps. I had to read it over a few times before I could really understand what was supposed to be proven. This confusion often felt very discouraging to me, but it was also one of my first experiences in reading and understanding historical math sources [Coco, OE4].
Orientated toward a meaningful change	The PSP (and abstract algebra) has helped cultivate a desire to ask better questions in mathematics and obtain better answers. My formulations of answers before this were rather sloppy and disorganized, however afterwards I almost demand more of myself when answering problems and don't allow for as many gaps in my arguments [Alton, OE1].
Accompanied by positive affective experiences, especially hope	I like it more than ever before, using primary source projects gives me a lot of insight into what the mathematicians were thinking and how they went about creating something completely new. When going through each "task" I get this small sense of accomplishment by solving proofs and making conjectures, giving a sense [of] discovery and fulfilment [Ryan, OE2].
Inner-directed; depend on the components of personality, for instance, creativity, knowledge, motivation, courage, perseverance	The PSP gave me more motivation to read and try to understand mathematical sources. While reading through the packet and working through each task, I enjoyed trying to understand and reach a conclusion on my own [Coco, OE2].
Possible	I feel that this approach allows me to understand the thought processes of the mathematicians that first discovered these ideas; in particular, I feel that I can discover some of these ideas on my own just by sharing some of their thought process [Lono, OE3].

Note. OE#: The open-ended item number (see **Table 1**)

possible with regard to transgressive triads in mathematics learning via PSPs. For instance, we discovered that transgressive actions are prevalent in such a learning experience, and we conjecture that similar transgressive actions are undertaken when students study mathematics via other non-traditional instructional methods. We have also proposed that transgressive actions connect boundaries and outcomes together within triads. The only caveat is that boundaries are difficult to discern. Such a phenomenon is not surprising when considering, for example, the change that occurs during psychological therapy (or, psychotherapy), allowing individuals to see that something that was not possible for them before has become achievable. This is a limitation for our own research regarding boundaries existing in mathematics education. Because of the close relationship between boundaries and outcomes, we predict that boundaries would be significantly difficult to study (while students are being bound by them) if we do not know the outcome before it is manifested. In this sense, a theoretical exploration grounded in data was necessary to develop a starting point with regard to boundaries reflected in non-traditional instruction methods in mathematics education. While we recognise that some boundaries around learning mathematics via PHS have been reported, our illustrative transgressive triad based in situated learning theory suggests that other boundaries may be historically ingrained in students due to traditional instructional practices they typically experience. If we expect students to make great leaps in overcoming historically ingrained boundaries, it stands to reason that traditional instruction cannot satisfy all the demands of escorting students into the mathematical community. Thus, studying mathematics via PSPs is one way students may transgress to gain access to the practices that typically only well-established members of the community experience.

Therefore, we envision a fruitful landscape for opportunities to build on the initial research conceptualisations we offered in this article. For example, we believe the undergraduate mathematics education research domain is an optimal context for situating investigations focused on not only how and why students undertake transgressive actions, but well-designed research will provide what still requires further study, including the detection of the boundaries students' transgressive actions enable them to cross and the outcomes students subsequently access. To this end, we have recently designed a new study for which a key data source is a sequence of interviews to capture students' perception of the triad components while learning mathematics via PHS. Scholars (e.g., Jahnke et al., 2000) have long asserted that the use of PHS has the ability to challenge students' perceptions and conceptions by making the familiar unfamiliar. We posit that this may in fact contribute to the natural fit that the transgressions lens lends to research that investigates students' actions granting them access to the community via the mathematical practices enabled by accomplices (i.e., instructional tools, curriculum materials, people). However, it is also relevant for other interventions which engage undergraduate mathematics students while making the familiar unfamiliar, such as research investigating the many perspectives on the use of active learning, the role of digital technologies in mathematical sense making, and the role of writing and discussion in mathematics learning spaces.

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