International Electronic Journal of Mathematics Education
2024, 19(4), em0797
e-ISSN: 1306-3030
https://www.iejme.com
MODESTUM
Research Article
OPEN ACCESS

Conceptualizing algebraic connections: Honoring voices from future mathematics teachers and the instructors who teach them

Hyunyi Jung [1*], Jill Newton [2], Eryn M. Maher [3]

[1] Texas A&M University, College Station, TX, USA
[2] Purdue University, West Lafayette, IN, USA
[3] Georgia Southern University, Statesboro, GA, USA
***Corresponding Author:** hyunyi.jung@tamu.edu

Citation: Jung, H., Newton, J., & Maher, E. M. (2024). Conceptualizing algebraic connections: Honoring voices from future mathematics teachers and the instructors who teach them. *International Electronic Journal of Mathematics Education, 19*(4), em0797. https://doi.org/10.29333/iejme/15519

ARTICLE INFO
Received: 14 Mar. 2024
Accepted: 26 Sep. 2024

ABSTRACT
Beginning teachers benefit from preparation emphasizing supporting students with diverse needs and developing deeper understandings of algebraic connections. Our study aims to explore how instructors in secondary preparation programs and mathematics preservice teachers (M-PSTs) conceptualize and enact algebraic connections in required courses. Our dataset comprises instructor interviews and course materials from 48 courses across five universities, and 10 focus group interviews involving 37 M-PSTs. Employing constant comparison method, we analyzed data by creating a coding system for algebraic connections. Our findings highlight varying perspectives of M-PSTs and instructors related to encounters with algebraic connections, emphasis of algebraic connection types by both groups, and types emphasized as main objectives of courses in five secondary mathematics teacher education programs. We describe nine algebraic connection themes that were identified in this investigation. We discuss emergent implications for curriculum development, instructional practices, and foundational algebraic concepts in teacher preparation programs.

Keywords: algebraic connections, algebra teaching and learning, mathematics teacher preparation, secondary mathematics preservice teachers

INTRODUCTION

Given the foundational necessity of algebra for advanced mathematics and its status as a gatekeeper to post-secondary education, algebra is recognized as a critical subject to which all students need access (Kilpatrick & Izsák, 2008; Rech & Harrington, 2000). Algebra classes include a more diverse population because of “algebra for all” movements (Stein et al., 2011; Teuscher et al., 2008). Researchers have recommended changes in how algebra is taught, suggesting that its teaching shift from a symbolic orientation emphasizing procedures to an orientation emphasizing student engagement with representing functions and problem-solving in contexts (Kieran, 2007; Usiskin, 2015). Henry et al. (2012) reported that algebra courses are likely to be taught by beginning teachers, and these teachers teach a more diverse population than ever before. Beginning teachers would benefit from preparation that focuses on supporting students with diverse needs in gaining a deeper understanding of algebra and making connections (National Council of Teachers of Mathematics [NCTM], 2000).

NCTM (2000) suggested three types of connections that contribute to students’ deeper understanding of mathematics:

(a) connections among mathematical topics,

(b) connections to other subjects, and

(c) connections to real world situations.

The Association of Mathematics Teacher Educators (AMTE, 2017) suggested that teachers need to investigate such connections in their mathematics learning to effectively integrate encounters with connections into pedagogical approaches to algebra instruction. In addition to the three types of connections recommended by NCTM, beginning teachers need to experience and build connections between school and college-level algebra (AMTE, 2017; Conference Board of the Mathematical Sciences [CBMS], 2012). In *mathematical education of teachers II*, CBMS (2012) proposed a series of courses for secondary mathematics preservice teachers (M-PSTs) that feature experiences to support making connections between secondary school algebraic topics and advanced college-level algebra topics.

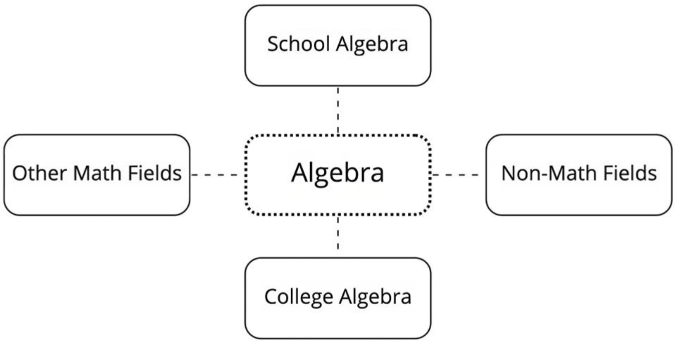

Figure 1. Our conceptualization of algebraic connections (Source: Authors' own elaboration)

Despite the emphasis on algebra and its importance for college and career readiness, mathematics educators know little about how mathematics teachers are being prepared to teach algebra in programs across the United States. To identify algebraic connections in teacher preparation programs and highlight potential areas of connections for mathematics teacher educators, we collected and categorized experiences reported by instructors of required courses and by M-PSTs in their final semester as supportive of making algebraic connections. Our research question is "How do instructors of required courses in secondary preparation programs and secondary M-PSTs conceptualize and enact making algebraic connections?"

RELEVANT LITERATURE

In this section, we first justify attention to M-PSTs' preparation to learn about algebra. We then discuss the necessity for M-PSTs to experience and learn about teaching connections in mathematics broadly and, more specifically, in algebra.

Algebra Learning and Its Connections to Teach Mathematics

Valued as a foundational subject in mathematics and other disciplines, algebra acts as a gatekeeper for students' access to post-secondary opportunities (Kilpatrick & Izsák, 2008; Moses & Cobb, 2001). In response to algebra-for-all initiatives, many states have revised high school graduation requirements to include successful completion of one or more algebra courses (Teuscher et al., 2008). Poor integration of procedural fluency and conceptual understanding in grades 7-12, however, has left students ill-prepared to use algebra as a tool in advanced mathematics or real-world applications (Kieran, 2013; Usiskin, 2015). Ideally, efforts would focus on strategies for teaching algebra that support accessibility without sacrificing depth or rigor. With changing reform initiatives and expanding diversity of algebra learners, mathematics educators and curriculum developers have focused on algebra as an essential content that connects to other aspects of mathematical learning (Kieran, 2013; Kilpatrick & Izsák, 2008; Moses & Cobb, 2001).

García-García and Dolores-Flores (2019) conceptualized mathematical connections as "a cognitive process through which a person makes a true relationship between two or more mathematical ideas, concepts, definitions, theorems, or meanings with each other" (p. 2021). NCTM (2000) highlighted the importance of mathematical connections by including "connections" as one of five process standards essential for all K-12 students; these process standards were instrumental in the development of the standards for mathematical practice published in *common core state standards for mathematics* (*CCSSM;* National Governors Association [NGA] & Council of Chief State School Officers [CCSSO], 2010). Students need to see the "interrelatedness of mathematical ideas," and "the notion that mathematical ideas are connected should permeate the school experience at all levels" (NCTM, p. 64). Students need experiences that support learning how mathematical ideas are connected and developing the habit of mind to look for connections and use multiple representations of mathematical ideas (Haciomeroglu, 2007). Mathematical connections are important because students can

(a) use connected ideas to support problem-solving,

(b) view new mathematical ideas as extensions of mathematics learned previously, and

(c) make use of common patterns and structures (NCTM, 2000).

Instructors in secondary mathematics teacher education programs must make connections visible to M-PSTs (CBMS, 2012; NCTM, 2012). AMTE's (2017) *standards for preparing teachers of mathematics* recommended that M-PSTs have experiences in both mathematics and mathematics methods courses that support making connections between content areas (e.g., algebra and geometry), between mathematical ideas and authentic contexts, and between high school-level and college-level mathematics.

Conceptual Framework: Algebraic Connections

Given calls for M-PSTs to understand mathematics as a coherent field, experiences that support building algebraic connections are essential (AMTE 2017; CBMS, 2012; Cuoco & Rotman, 2013; McCrory et al., 2012). Based on recommendations from policy documents (AMTE, 2017; CBMS, 2012; NCTM 2000, 2012; NGA & CCSSO, 2010) and relevant research (Artzt et al., 2012; Carraher & Schliemann, 2007; Cuoco & Rotman, 2013; Foley, 1997; Gravemeijer, 2002; Kirshner, 2001; Lehrer & Schauble, 2000; McCrory et al., 2012; Moore-Harris, 1998; Usiskin et al., 2002), we describe our conceptual framework which highlights four types of algebraic connections recommended for M-PSTs to encounter in their secondary mathematics teacher education programs (see **Figure 1**).

We align our conceptual framework with prior research focused on vertical and horizontal curricular knowledge (Shulman, 1986; Tyler, 1949) and knowledge for algebra teaching (KAT) (McCrory et al., 2012). Shulman (1986) introduced vertical knowledge as a familiarity with specific topics and materials that "have been and will be taught in the same subject areas" (p. 10). In our framework (**Figure 1**), vertical knowledge concerns connections between algebra and high school-level mathematics, as well as connections between algebra and college-level mathematics. These connections will hereafter be called "school and college-level connections." Lateral (horizontal) knowledge is conceptualized as familiarity with specific topics in a subject and its related topics in different subject areas. In our framework, horizontal knowledge involves connections between algebra and other mathematical fields (hereafter called "other math fields connections"), as well as connections between algebra and non-mathematical fields (hereafter called "non-math fields connections"). Our conceptualization of horizontal and vertical knowledge in algebra learning is also aligned with the KAT framework (McCrory et al., 2012) in which the authors emphasized *bridging*, which can be implemented in mathematics classrooms by "providing students with the big picture of mathematics, making explicit connections across topics, keeping a range of ideas in play in the classroom, and presenting mathematics as a coherent, connected endeavor" (p. 608). Building on this prior work, we focus on four types of algebraic connections:

(a) within algebra,

(b) school and college-level,

(c) other math fields, and

(d) non-math fields.

Within algebra connections

NCTM (2000) recommended secondary students explore connections within algebra, including those between algebraic concepts and representations. To facilitate such experiences, mathematics teachers use their understanding of connections to create experiences to make them visible. For example, connecting concepts of linear function, constant rate of change, slope, proportionality, and arithmetic sequences is important to support students in building on prior knowledge to enhance their understanding of subsequent mathematical concepts. McCrory et al. (2012) argued that algebra has not been a focus of teacher education programs, and given its importance and prevalence M-PSTs need support in developing their own conceptually rich and connected algebraic knowledge. The authors recommended that teacher preparation programs provide experiences that help future algebra teachers understand how algebraic concepts across mathematics courses connect to present algebra as a "connected, coherent domain" to their students (p. 608).

School and college-level connections

Foley (1997) recommended that M-PSTs learn about connections between high school algebra content and abstract topics taught in advanced courses such as linear algebra, abstract algebra, and real analysis. For instance, mathematics teachers need an understanding of "groups, rings, fields, and the associated theory. They need to recognize the importance of the complex numbers as a field and the significance of the fact that matrix multiplication is noncommutative" (p. 88). In support of Foley's recommendation, CBMS (2012) proposed courses in which M-PSTs make connections between advanced mathematics and school mathematics. Despite repeated calls for such courses, results from a national survey found that the majority of secondary mathematics teacher education programs surveyed were not offering these types of courses (Newton et al., 2014). AMTE (2017) referenced recommendations provided in CBMS (2012) and further argued that M-PSTs need to experience examples and connections in both standard mathematics courses and methods courses. Efforts have been made to develop curricula and textbooks for use in these courses (Bremigan et al., 2011; Cuoco & Rotman, 2013; Sultan & Artzt, 2010; Usiskin et al., 2002; Weiss, 2021).

Other math fields connections

M-PSTs need to experience algebra as intertwined throughout the discipline of mathematics. NCTM (2000) recommended that students be prepared to "recognize how mathematical ideas interconnect and build on one another to produce a coherent whole" (p. 64) across their PK-12 mathematics coursework. To prepare mathematics teachers to provide these experiences for their students, CBMS (2012) proposed teacher preparation programs provide experiences in mathematics courses for M-PSTs to investigate conceptual connections and explore connections within structures of mathematics. AMTE (2017) also urged teacher educators to support M-PSTs by making connections to algebra in their study of other mathematical content areas such as geometry and statistics. The National Research Council (2010) similarly argued that teachers need a coherent view "not just of the content they are responsible for teaching, but also of the broader mathematical context for that knowledge and the connections between the material they teach and other important mathematics content" (pp. 114-115).

Non-math fields connections

Moore-Harris (1998) argued that programs in mathematics become more accessible if students' experiences with algebra "are steeped in realistic and relevant contexts" (p. 46). Carraher and Schliemann (2007) presented the referential role of algebra, emphasizing that algebraic knowledge grows out of representing extra-mathematical contexts (Gravemeijer, 2002; Kirshner, 2001; Lehrer & Schauble, 2000). In support of these arguments, NCTM (2000) and AMTE (2017) emphasized that students need opportunities to recognize and apply mathematics in contexts outside of mathematics; in particular, making use of algebra to model and predict real world phenomena. NCTM (2009) stated that "connections between mathematics and real-world problems developed in mathematical modeling add value to, and provide incentive and context for, studying mathematical topics" (p. 13). AMTE (2017) emphasized pragmatist perspectives, stating that mathematics teachers are tasked with effectively exemplifying the critical use of mathematics in decision-making. Further, the *CCSSM* authors committed a standard for mathematical practice to

Table 1. Characteristics of universities

Characteristic	GLU	MRU	MUU	SRU	WUU
Basic Carnegie classification	Master's colleges & universities: Larger programs	Doctoral universities: Highest research activity	Master's colleges & universities: Larger programs	Doctoral universities: Highest research activity	Master's colleges & universities: Larger programs
Degree-seeking undergraduate race/ethnicity enrollment percentages[a]	2% Asian 5% Black 4% Latin@[a] 84% White 3% Multiracial	4% Asian 4% Black 5% Latin@ 71% White 4% Multiracial	10% Asian 10% Black 35% Latin@ 37% White 2% Multiracial	9% Asian 7% Black 5% Latin@ 73% White 3% Multiracial	16% Asian 5% Black 59% Latin@ 8% White 2% Multiracial

Note. GLU: Great Lakes University; MRU: Midwestern Research University; MUU: Midwestern Urban University; SRU: Southeastern Research University; WUU: Western Urban University; & [a]We use the @ sign to include all gender identifications

Table 2. Number of instructor interviews and M-PSTs interviewed at each university by course type

Course type	GLU	MRU	MUU	SRU	WUU	Total
Number of instructor interviews						
Mathematics	5	4	6	5	0	20
Mathematics for teachers	1	2	0	3	0	6
Mathematics education	3	5	3	4	1	16
General education	1	1	1	1	2	6
Total number of interviews	10	12	10	13	3	48
Number of M-PSTs interviewed	6	8	8	8	7	37

Note. GLU: Great Lakes University; MRU: Midwestern Research University; MUU: Midwestern Urban University; SRU: Southeastern Research University; & WUU: Western Urban University

modeling, including an entire secondary mathematics strand focused on specific modeling experiences using targeted content (NGA & CCSSO, 2010).

METHODS

Using the conceptual framework of the four types of algebraic connections described above, we investigated how instructors of required courses in secondary preparation programs and M-PSTs conceptualized and enacted algebraic connections. We utilized instructor interviews and course materials from 48 courses across five universities and 10 focus group interviews with 37 M-PSTs. We analyzed experiences reported in interviews from instructors and M-PSTs, and written course materials (e.g., syllabi and assignments). We triangulated our data through the use of instructors' reports of intended and implemented algebraic connections, written materials, and interviews with M-PSTs near the end of their program about the algebraic connections.

Programs and Participants

The Preparing to Teach Algebra (PTA) research team utilized a mixed methods approach with a national survey and case studies; this paper focuses on case study data from the PTA project. We selected programs at five universities for further examination based on their geographic location, the nature of their student populations, the types of communities, and the departmental homes of their secondary mathematics teacher education programs (e.g., college of education, mathematics department), not in an effort to compare them or provide a representative sample, but rather to provide experiences from a range of university contexts. **Table 1** provides a summary of the characteristics of the five programs.

At each case study site, members of the research team conducted interviews with instructors of approximately 10 required courses and collected their associated instructional materials. The background of the instructors interviewed for the study varied in terms of position, years working at the university, and terminal degree. Across universities, 41% were full professors, 20% were associate professors, and 23% were assistant professors. The remaining 16% were adjunct professors, lecturers, and graduate students. Forty-four percent of the instructors reported working at the university for 10 years or less; 40% between 11 and 20 years, and 17% more than 20 years. We did not ask the instructors to report their gender and racial/ethnic background.

With the help of a site coordinator familiar with the program, we selected courses based on their potential to include experiences to learn algebra, to learn to teach algebra, and/to to learn about equity in algebra. The courses were of four main types: mathematics (e.g., linear algebra), mathematics-for-teachers (e.g., algebra for teachers), mathematics education (e.g., secondary mathematics methods), and general education courses with a focus on educational equity (e.g., teaching in a diverse society). For confidentiality and to highlight similar courses, we used generic course titles. WUU's largest program was a post-baccalaureate program for students who already had a mathematics major and only offered mathematics education and general education courses; thus, interviews for mathematics or mathematics-for-teachers courses were not conducted. **Table 2** summarizes, by course type, the instructor interviews conducted at each university.

As **Table 2** indicates, most instructors interviewed taught mathematics or mathematics education courses rather than mathematics courses designed for M-PSTs. The greatest number of mathematics education courses was at MRU. Two programs did not offer a mathematics-for-teachers course, while SRU offered a sequence of three mathematics-for-teachers courses.

Table 3. Sample algebraic connections

Type of algebra connection	Coded interview quote or statement from syllabi
Within algebra	*Interviewer:* [Showing the four connection types] Which of these types of connections do you emphasize in your course? *Reasoning & proof instructor at MRU:* [pointing out within algebra connection]"… I do a little bit of elementary group theory, and I also do elementary number theory. And you can use elementary group theory to prove things in number theory and, of course, you can use number theory to prove things in group theory, so there's lots of interplay there."
School and college-level	*Interviewer:* Which problems or activities do you use to help your students make school and college-level connection? *Abstract algebra instructor at MUU:* "… I have them think about what it means to solve for *x* in an equation they're familiar with, which is from something they're familiar with previous math classes and then going to the abstract notion of what we mean by a group."
Other math fields	*Course objective of the geometry syllabus from GLU:* "This course is a critical analysis of Euclidean geometry from transformational, algebraic, and synthetic perspectives in two and three dimensions."
Non-math fields	*Interviewer:* Which problems or activities do you use to help your students make connection between algebra and non-math? *Secondary mathematics methods instructor at WUU:* "In terms of re-districting for congressional, we talk about area. We talk about distribution. Is that fair distribution? And depending on the case. I mean it's an election time. Normally we talk about elections."

At each site, we also conducted two focus group interviews each with 3-4 M-PSTs who were near the end of their program; **Table 2** shows the number of M-PSTs at each site. The focus group goals included gaining an overall perspective about their secondary mathematics education programs. In addition, we asked M-PSTs to reflect individually and then discuss their recollections of experiences with algebraic connections in their coursework.

Data Collection

We collected data from three sources: instructor interviews, corresponding instructional materials, and M-PST focus group interviews. Our framework, based on the policy documents and related research described above, included four major types of algebraic connections:

(a) within algebra,

(b) school and college-level,

(c) other math fields, and

(d) non-math fields.

We explicitly asked both instructors and M-PSTs about their experiences with these connections, providing them with a document describing the four types. We asked the instructors to answer questions about algebraic connections in their course:

(1) Which of these types of connections do you emphasize in your course?

(2) Which problems or activities do you use to help your students make these connections?

The parallel questions for M-PSTs, asking them to consider coursework across their program, were:

(1) How much does your program emphasize these connections?

(2) Which specific problems or activities helped you learn about these connections?

Because algebra is a complex discipline, algebraic ideas can be viewed from many perspectives. We were interested in the activities that instructors themselves perceived as algebraic, so we included the following response in our interview protocol to use if asked for a definition, "I recognize there are many views of algebra. Please use your own conceptions of algebra, as it is used in your course."

Data Analysis

Our analysis involved three phases:

(1) developing and utilizing a coding system for the larger PTA study,

(2) creating and using a coding system for algebraic connections, and

(3) summarizing data for our research question.

For the first phase, the research team identified all experiences required for every M-PST in the class *and* identified by the instructor and/or M-PST as algebraic in nature (i.e., they described the experience when asked for opportunities to learn algebra). Second, two researchers coded each experience according to the four types of algebraic connections (see **Table 3** for sample algebraic connections). The interrater reliability for the initial coding of each experience ranged from moderate agreement (k = 0.48) to perfect agreement (k = 1.00) (Landis & Koch, 1977). Pairs of researchers met to resolve each discrepancy, assign the consensus code, and document coded data on which both researchers agreed.

These types of algebraic connections are not discrete; for example, an instructor might describe an experience in which M-PSTs model real world data (non-math fields connection) to see other math fields connections. Therefore, even though most activities were coded as one type of connection, in a few cases we coded the activity as two connection types when both connections were explicitly discussed. We then produced frequencies for algebraic connections across programs and by

Table 4. Total number of algebraic connections reported by all instructors and M-PSTs

Type of algebra connection	Reported by instructors	Reported by M-PSTs	Total
Within algebra	30 (24%)	5 (14%)	35
School and college-Level	18 (14%)	7 (20%)	25
Other math fields	44 (35%)	6 (17%)	50
Non-math fields	33 (26%)	17 (49%)	50
Total	125 (100%)	35 (100%)	160

instructors and M-PSTs. We also examined interview transcripts to draw themes around the conceptualizations of algebraic connections. Using the constant comparison method (Glaser, 1965), each of us reviewed and summarized interview transcripts, and discussed emergent themes. As we iteratively reviewed both the themes and interview transcripts, we reached a consensus on the emergent themes within each type of algebraic connection.

FINDINGS

We first present the number of algebraic connections reported by the two groups of participants, course instructors and M-PSTs. We then describe the ways that each group conceptualized the four types of algebraic connections:

(a) within algebra,

(b) school and college-level,

(c) other math fields, and

(d) non-math fields.

Overview: Algebraic Connections Reported by Instructors and M-PSTs

In **Table 4**, we present an overview of the number of algebraic connections reported by instructors and M-PSTs during the interviews at each of the five universities.

Overall, other math fields and non-math fields connections were reported more frequently than the other two types of connections in the five teacher education programs. When comparing connections reported by instructors versus M-PSTs, other math fields connections were most often reported by instructors, whereas M-PSTs described more non-math fields connections. Instructors reported school and college-level connections much less often, whereas M-PSTs reported within algebra, other math fields, and school and college-level connections in similar numbers. In the following sections, we share the themes which emerged from conversations with instructors and M-PSTs.

Within Algebra Connections

Nearly 25% of the algebraic connections reported by instructors highlighted connections within algebra. Abstract algebra and linear algebra instructors at all four universities that offer these courses mentioned at least one within algebra connection. For example, the abstract algebra instructor at MUU shared: “We talk about the division algorithm and incorporating that into proofs; some more advanced, abstract proofs, they have to incorporate ideas that they’re familiar with from college algebra and think about how these number systems relate to each other.” Likewise, at least one within algebra connection was mentioned by secondary mathematics methods instructors at all five universities. For example, one methods instructor at MUU stated: “I would say when we talk within algebra, we have students make connections among different representations, so, graphs, tables, words, and symbols.” Three of the programs (i.e., GLU, MRU, and MUU) required at least one course that included a within algebra connection as a big idea. Two themes emerged from a review of all within algebra connections:

(a) algebra as a complex web of concepts, objects, processes, and structures and

(b) algebra as representations.

Algebra as a complex web of concepts, objects, processes, and structures

Instructors at all five universities described making connections within algebra in their courses, often alluding to the complex relationships between algebraic concepts, objects, processes, and structures, and even, at times, identified these connections as the essence of the course: “I cannot have [identify] connections within algebra, that’s all it is, is just connections amongst itself” (linear algebra, SRU). As we examined the instructors’ statements, a web of connections emerged. For example, several instructors described connections to polynomials:

(a) an abstract algebra instructor at SRU mentioned making connections between integers and polynomials,

(b) a secondary mathematics methods instructor at WUU described emphasizing the extension of binomials to polynomials, and

(c) an algebra for teachers instructor at MRU introduced his students to Galois’ revolutionary theory in which he connected the idea of a group of permuting roots of a polynomial to field theory.

These instructors included attention to polynomials connected to integers, binomials, and field theory, representing one small cluster of the web of “within algebra” connections.

We noticed two, often overlapping, ways that instructors described components of this algebraic web:

(a) as overarching big ideas that included other topics and

(b) as foundational big ideas that build up to or support other topics.

Two instructors at SRU described foundational big ideas related to reasoning. The secondary mathematics connections I instructor proposed algebraic reasoning as foundational, stating:

> If we define it [algebra] just in terms of algebraic reasoning, that's where the connections are … Like, what type of reasoning would we want high school students to do, engage in, that will give them a foundation for formal algebra and moving into structures and abstractions, into kind of abstract structures?

The secondary mathematics methods III instructor described multiplicative reasoning as "the foundation for thinking about other algebraic concepts."

The abstract algebra instructors at GLU and MUU advocated for attention to number systems to connect algebraic ideas. The GLU instructor promoted "looking at common structures and themes behind different number systems and the way algebra works in those number systems" and the MUU instructor emphasized "a connection to their prior understanding of algebra because some of the groups we study are familiar number systems like integers, reals, rationales, etc. So, they use their familiar algebraic properties as they study group theory." At these same universities, mathematics education instructors also highlighted algebraic connections related to properties. The GLU secondary mathematics methods instructor mentioned connecting properties (i.e., distributivity, associativity) to solving equations and the MUU student teaching seminar instructor stated:

> One of the things that I like to do when I teach math in general is make connections. It's like there's an identity element for addition. There's an identity element for multiplication. There's an identity element in transformations of various types.

Instructors also identified multiple "key" algebra concepts in their courses, including finite groups, subgroups, cyclic groups, permutations, isomorphisms (abstract algebra, MUU) and functions, ordinary manipulations, linearity (probability & statistics, MRU). The student teaching seminar instructor at GLU also explicitly described his discussions with M-PSTs about connections within algebra in terms of their students' learning, saying:

> So, they'll [M-PSTs] see a student who is struggling with one thing in algebra and realize that they have under-developed pre-requisite knowledge. They'll be solving polynomial equations and realize the students don't have much of an understanding of exponents … that's preventing them from being able to understand function applications. If we can track back student difficulties to the underlying concept, those tend to be exceptionally productive discussions because now they [M-PSTs] really care … they're very motivated and have all this kind of data about it.

These within algebra connections highlighted by instructors and M-PSTs illustrate the common structures in algebraic systems and both the necessity of making these connections explicit (e.g., looking back and looking ahead) and the potential learning gains for students who have had opportunities to investigate these connections.

Algebra as representations

Instructors and M-PSTs often pointed to relationships among representations as a type of within algebra connection. Instructors' examples of representations included graphs, tables, symbols, words, and physical objects. The calculus in the curriculum instructor at MRU referenced the "rule of four" (i.e., presenting algebra geometrically, numerically, analytically, and verbally) and the secondary mathematics methods instructor at MUU emphasized the importance of opportunities for M-PSTs to "translate between representations." Instructors discussed the importance of M-PSTs' experiences with representations in foundational or overarching ways. The foundational perspective emerged when instructors described how their students build on their knowledge of certain representations to understand new concepts (e.g., mathematics for secondary teachers, GLU; secondary mathematics methods, MUU). In this sense the representations were used in a foundational way, providing support to build on M-PSTs' understanding of concrete or visual representations to make sense of or construct more abstract or complex representations. Other instructors described representations in ways that seemed overarching for facilitating connections within algebra. The mathematics for secondary teachers instructor at GLU, for example, described algebra as the "written language of mathematics" and suggested that symbolic representations of ideas or relationships are tools used to connect algebraic ideas. Another instructor emphasized that "symbols always symbolize something, they're not just marks on the paper" (secondary mathematics connections I, SRU).

Some descriptions touched on both foundational and overarching perspectives. The secondary mathematics methods instructor at MUU described using concrete representations of "tiles surrounding a pool" to support students in constructing symbolic generalizations of the situation. In this sense, the description of representations seemed foundational. The instructor went on to explain that because students notice different patterns in the visual representation, they generate different forms of equivalent symbolic expressions. The class then examined the patterns and their respective expressions to discover that the expressions were equivalent and to notice how the parts of each expression relate to a slightly different perspective. In this way, the connections between algebraic representations offered the potential to guide M-PSTs to see overarching algebraic structures.

Both instructors and M-PSTs mentioned the role of representations within algebra in relation to teaching secondary mathematics. For example, the middle school mathematics methods instructor at GLU emphasized how connections between representations and students' different approaches to solving problems promote building on one another's ideas and strategies. The pedagogical aspects of teaching about multiple representations were addressed in both the algebra in the curriculum and modeling in the curriculum courses at MRU. The algebra in the curriculum syllabus included the objective "Develop insights about teaching the relationships between equations and their graphs" and the modeling in the curriculum instructor highlighted the importance of comparing representations of linear, quadratic and exponential models, including attention to the impact and meaning of parameters in each situation. Two M-PSTs at MRU reminisced about exploring representations in these courses; one stated, "I remember when I was in algebra, I always thought of equations and graphs as very separate things, and I think that's a big part of modeling, is starting to think of them as one thing." An M-PST at SRU reported experiences in secondary mathematics methods III that helped her see the importance of "providing students with tasks so that they can make connections."

As recommended by McCrory et al. (2012), the instructors and M-PSTs recognized the important connections within the domain of algebra. The instructors discussed their efforts to make connections among algebraic concepts, objects, processes, and structures explicit in their courses and to highlight the pedagogical implications of the connections. Instructors alluded to both the foundational and overarching aspects of the web of algebraic ideas and the role of representations in algebra.

School and College-Level Connections

High school-level algebra and college-level algebra involve connected reasoning and content. For example, both address algebraic thinking which encompasses studying relationships, generalizations, and analyzing change. School algebra includes more concrete forms of algebraic thinking while college-level algebra moves deeper into more abstract forms. Both school-level algebra and college-level algebra support development of problem-solving skills, explore multiple representations, and work with variables, operations, and equivalent equations. School-level algebra is primarily focused on solving equations and graphing functions based on real-world situations. In contrast, college-level algebra explores the underlying principles and abstract structures; for example, considering how changes in elements and operations impact equivalence relations and properties of groups or rings. Two themes that describe connections between school and college-level algebra were made visible through our collected reports from instructors and M-PSTs:

(a) teaching high school algebra to support students' future experiences and

(b) building on school algebra to teach college-level algebra.

Teaching high school algebra to support students' future experiences

Course instructors and M-PSTs described how making connections between school and college-level algebra can support students' future experiences in several ways. M-PSTs described how algebraic concepts and procedures work, how specific problems and concepts in college-level algebra build on and extend certain problems and concepts in high school, and how they might motivate and empathize with their future students. M-PSTs in focus groups at GLU, MUU, and SRU reported that connections between school and college-level algebra helped them understand how and why certain algebraic concepts and procedures work. One M-PST at GLU described connecting the how with the why: "[I]t really helped me be able to express why we do things so whenever students ask me questions–Why?'–I can explain why certain things make sense … Really getting at the why." Another M-PST at GLU described getting to see the "behind-the-scenes stuff" and gaining a new perspective of seeing "why everything works:"

> There's a lot of behind-the-scenes stuff that we do in high school that nobody ever talks about. We do it because you just do it … and I think it's interesting now seeing those connections … rather than just be told 'It works because that's just the way math is. Don't worry about it … it's good to be at this point … here I see a lot of why everything works.

Describing a specific example of connecting the how and the why, an M-PST at MUU explained:

> We did proofs on whether or not two numbers must have a relationship of more than or less than each other … I remember that taking for granted that when you divide by a negative number, the inequality flips … if you understand it as part of a definition you don't just do it at the end, like 'Oh, this is just … the rule,' but you actually understand that this is how it's supposed to be because of the definition.

This future teacher described how they noticed their "students writing it the wrong way and then memorizing it as a rule" and were able to help the students see the "why" instead of just accepting the rule.

M-PSTs at GLU, MUU, and SRU reported that connections between school and college-level algebra helped them understand how specific problems and concepts in college-level algebra build on and extend certain problems and concepts in high school. Instructors at GLU, MRU, and SRU reported how they integrated such connections into their courses, including secondary mathematics methods courses as well as mathematics content courses (i.e., reasoning and proof, abstract algebra, and probability and statistics). The abstract algebra instructor at GLU and the probability and statistics instructor at SRU described examples of K-12 algebraic thinking that they used to explicitly connect and build on in their courses. For example, the GLU abstract algebra instructor reported:

> We talk about the relationships between [polynomial] problems that we might ask at this level and the corresponding problems at the high school level. And the same thing with the integers … and one of the questions I'll ask is …' how might a fourth grader divide 43 by 5?' … We talk about, 'Well, they might start taking away fives, until they stop.' And so that kind of motivates some of the machinery in the division algorithm, how you're looking for a remainder that's less than the divisor, and things of that sort.

The three "in the curriculum" (i.e., modeling, algebra, and calculus) course instructors at MRU and the secondary mathematics connections instructor at SRU described school and college-level algebra connections throughout their courses. For example, the secondary mathematics connections I instructor explained an explicit emphasis on answering the question "What type of reasoning would we want high school students [to be] engaging in, that will give them a foundation for formal algebra and moving into [algebraic] structures and abstractions?" At MRU, the algebra in the curriculum instructor explained how he tried to bring in high school problems and concepts that related to topics in the M-PSTs' linear algebra course (taken concurrently).

M-PSTs at SRU reported that connections between school and college-level algebra helped them understand how they might motivate and empathize with their future students. Instructors at GLU and SRU reported how they saw such motivation and empathy in their courses. For example, the student teaching seminar instructor at GLU explained:

> Sometimes you'll hear them trying to justify school algebra to their students, in terms of what they'll need when they go to college. Or they'll kind of connect it to experiences they've had like the algebra that's necessary for use in calculus.

M-PSTs at SRU also explained how the methods and connections courses helped them reflect on how they had learned certain concepts and how that could inform their teaching. Both instructors and M-PSTs discussed how further experiences with high school algebra topics during their program better prepared M-PSTs for their advanced mathematics classes, helped them develop a deeper understanding of the high school topics, and gave them the tools to make connections between school-level and college-level algebra for their future students.

Building on school algebra to teach college-level algebra

At every university in the study, M-PSTs and instructors of both mathematics content and mathematics methods courses mentioned the importance of proficiency in college-level algebra topics of algebraic notation, terminology, and manipulations learned in high school. The Discrete Mathematics instructor at MUU explained that connecting school and college-level algebra fits into a large perspective of expanding their concept of algebra. He said:

> We don't necessarily want to get rid of anything that they have brought in, [for example,] if the idea of 'a function is something with a rule' serves a purpose for them in their classes, we don't want to necessarily dismiss it, but we want to enlarge that, so they understand the more modern formulation of functions and how their previous understanding fits into that.

Several M-PSTs and instructors at each university similarly mentioned how college-level algebra builds on and extends algebraic reasoning from K-12, especially how ideas of equivalence and equals, representations, properties, and rules are expanded or are similar and different across, for example, number systems, matrices, or geometries.

A specific strategy mentioned by M-PSTs and instructors at each university was deepening algebraic understanding through proofs and formal reasoning. To prove algebraic concepts using connections between school and college-level algebra, M-PSTs reported applying the principles and techniques they learned in K-12 and building on their understanding of algebraic manipulations, properties of equality and operations, and utility of multiple representations. For example, an M-PST at GLU reported: "I saw that [extending high school algebra] in a lot of our proof classes as well where they would take something that we learned in high school algebra, and we would twist it and try to prove it with college-level math." An MUU M-PST summarized: "Understanding [a math topic], delving deeper into it, and re-learning it from a proof way." Overall, the instructors and M-PSTs at all five universities echoed the sentiment from CBMS (2012) that "building theories directly connected to high school mathematics can also strengthen and deepen prospective teachers' knowledge of what they will teach" (p. 56).

Other Math Fields Connections

In U.S. high school and university courses, mathematical content is often separated by mathematical field. For example, many high school students take an algebra I, geometry, algebra II course sequence; this structure does little to promote making connections between algebra and other mathematical fields. Despite this challenge, the instructors interviewed for this study described making connections between algebra and other mathematical fields more often than the other three types of connections. In contrast, this type of connection was less often mentioned by the M-PSTs. When M-PSTs did mention these connections, their descriptions were quite general. For example, an M-PST at MUU stated, "Algebra is so universal, I think, with all these math classes that it's just a continuation of using it. You're going to have variables in everything, they are going to continue to use them." This broad approach to describing these connections was also common among instructors; however, some instructors also mentioned specific examples of Other Math Fields connections. For example, an MRU Probability and Statistics instructor explained, "I mean they really have to understand the idea, the connection between areas under graphs and integrals. That's absolutely crucial for our discussion of continuous random variables…that's a very important connection that is necessary for success in the course." The themes which emerged from a review of the connections between algebra and other mathematical fields highlighted by instructors were:

(a) algebraic representations and structures across mathematical domains and

(b) relationships to algebraic objects and concepts in other mathematical fields.

Algebraic representations and structures across mathematical domains

Instructors often highlighted the connections between algebra and other mathematical fields by describing connections to algebraic structures and representations in other mathematical domains. At GLU, the abstract algebra syllabus stated: "Our investigations will begin with the integers and end with polynomials. Along the way, we will study the common algebraic structures that characterize these and other number systems," highlighting the common structures used in arithmetic and algebra. Two instructors at MRU echoed the arithmetic-algebra connection. The secondary mathematics methods II instructor pointed out that "seeing the parallels between algebraic and arithmetic solution methods can help students see that algebra is not something totally new but a more powerful tool to approach problems that are too hard to approach using arithmetic alone," and the algebra in the curriculum instructor emphasized that "the connection between work with computation and how work with arithmetic and computation can be generalized to help lay the foundation for algebraic thinking and for generalization for formalization." The MRU geometry for teachers instructor highlighted an algebra-geometry connection, saying:

> Between algebra and geometry, that was a very important connection that I constantly exploited. For example, in my computations with geometric entities, I use rules of algebra, like formulas from algebra also the trigonometric functions we have. So let's say, half of the course was an illustration of how algebra may be used with geometry.

The GLU capstone instructor explained algebra's role in enabling communication within and across algebra, including discussions with M-PSTs about algebra as a language:

> That algebra is this way to symbolically represent mathematical ideas is something that is pervasive in discrete mathematics, in geometry. It's in probability and statistics. It's in calculus and analysis, so in many ways algebra is this language. That algebra is the language of mathematics, right? On the mid-term exam, the essay question that I asked of students is do you agree or disagree with this statement: "Mathematics is a language." And the students were about a split, 50-50 … some students said that is the whole point of mathematics. That it is a language that … allows us to express ideas through that.

The mathematics for secondary teachers instructor at GLU echoed the idea of symbolic algebra as a language across mathematical fields:

> Between algebra and other mathematical fields, we connect to geometry, probability, measurement, statistics. One of the conceptions of algebra that we talk about is algebra, especially symbolic algebra, as a kind of the written language of mathematics and how being able to make a symbolic representation of an idea or a relationship gives us access to so many other tools, that's a powerful problem solving tool.

Instructors at several universities emphasized algebraic representations and symbols as an important way in which algebra connects to other mathematical fields, emphasizing how the consistency in the "language of mathematics" enhances these connections.

Relationships to algebraic objects and concepts in other math fields

Instructors reported including connections to algebraic ideas in other mathematical fields in their courses. The MRU linear algebra instructor explained his efforts to connect algebra to calculus:

> I try to talk about how it [algebra] is related to calculus in the sense that when you're doing calculus, you're essentially approximating a graph by its tangent line, finding its tangent line, and then you use that to approximate and that generalizes to a tangent plane, of a surface, or tangent space of a manifold … that allows you to linearize and in order to understand what happens once you've linearized, you need linear algebra to sort out what are intersections between the vector spaces and so on.

Several instructors also highlighted relationships between algebraic and geometric topics. For example, the abstract algebra syllabi at MUU included an objective that made this connection explicit: "Identify when algebraic ideas can be used to study objects and concepts in other mathematical fields, especially geometry." The GLU geometry instructor explained that M-PSTs need to know about connections between algebra and geometry "because of the relations of all those geometrical figures and rules and we are using once in a while this college-level algebra: the group, ring, and those type of things in the compositions. Also connecting algebra to geometry, the discrete mathematics instructor at MUU discussed the relationships between equality and both isometry and congruence:

> If you're in geometry class, equals mean isometry and what you're doing is you're paying attention to the geometric aspects. You're using that lens … and so we talk about equality of rational numbers versus equality of integers versus talking to them about … I talk about when you talk about congruence, congruent triangles or geometric figures, that's an equals. What are the properties of that congruence? It's reflexive, symmetric, and transitive. And what does defining that equivalence give you? New objects, equivalence classes to work with.

These algebra-geometry connections were also highlighted by mathematics education instructors. For example, instructors in secondary mathematics methods courses at MRU and MUU both mentioned the importance of algebra-geometry connections. The MRU instructor explained that: "Geometric representations such as graphs or figures can cast light on algebraic expressions and equations, and algebraic representations can be used to deduce geometric relationships." The MRU instructor provided several examples of experiences in which M-PSTs connected algebra and geometry.

> Like the border problem would be an example of that, the staircase problem. Using and, I would say, probably even algebra tiles, using them to model trinomials, that certainly is based on a geometric representation of trinomials.

As recommended by CBMS (2012) and AMTE (2017), instructors in these mathematics and mathematics education courses reported conceptual and structural algebraic connections to other mathematical fields, including arithmetic, calculus, geometry, probability, and statistics.

Non-Math Fields Connections

In their interviews, reports from both instructors and M-PSTs alike demonstrated that they valued connections between algebra and non-mathematical fields. Of the four types of algebra connections, non-math fields connections was the second-most-often-mentioned by instructors (26%) and the most often mentioned by M-PSTs (49%). An example of non-math fields connections commonly shared by instructors and M-PSTs was to bring a real-world context to algebra learning. Both groups shared multiple connections between real-world contexts and algebra, such as examining patterns in tiling a pool and examining patterns of change in gas prices. Mathematics instructors who taught courses without an algebraic focus reported drawing on their students' knowledge of algebra to analyze and understand real-world situations. Similarly, M-PSTs reported using algebra concepts to help their students make sense of situations in their lives. Three themes emerged from a review of all non-math fields connections:

(a) real-world connections used to illuminate algebraic concepts,

(b) algebraic connections used to provide insight into real-world situations, and

(c) real-world connections used to motivate interest in algebra.

Real-world connections used to illuminate algebraic concepts

By considering real-life situations (tangible, meaningful, relevant, and authentic), learning opportunities arise to support students who otherwise struggle to find meaning for algebraic concepts. Both instructors and M-PSTs described learning opportunities in which M-PSTs were introduced to algebraic concepts through a real-world context. For example, the secondary mathematics methods instructor at MUU described using concrete real-world contexts such as the Border Problem (tiling a rectangular pool) to support M-PSTs in realizing how different visualizations of a situation can result in different, but equivalent, algebraic expressions. In this activity, M-PSTs evaluated equivalence of expressions through symbolic manipulation and redrawing visual representations. Secondary mathematics courses explicitly explore and highlight real-world numerical relationships that can support algebraic understanding. A GLU secondary mathematics methods instructor noticed that some M-PSTs were struggling to understand systems of linear equations and used a gas station context as a way to demonstrate where and how systems of linear equations could be used.

Linear algebra courses present many examples of using non-mathematical contexts to understand algebraic concepts more deeply. For example, linear algebra instructors mentioned using population dynamics in ecology to introduce models of dynamical systems (GLU), using the computer scientists' challenge of identifying the most computationally effective strategies to understand orthogonal bases (MRU) or using contexts from economics and ecology to derive and make sense of the formula for Fibonacci numbers (SRU). In these non-mathematical contexts, the algebraic concepts are highlighted, providing opportunities to enhance M-PSTs' understanding of the concepts.

Algebraic connections used to provide insight into real-world situations

Instructors and M-PSTs also applied algebra as a tool to analyze, provide insight into, and solve problems in real-world situations. Probability and Statistics instructors mentioned applying algebra with statistical regression to understand situations in biology or the social sciences (GLU), to understand quality and process control or political attitudes (MRU), and to apply waiting period calculations to understand resource management (SRU).

An MRU mathematical modeling instructor referenced typical real-world examples as being simplified, adapted to be more precise, in an effort to support specific algebraic concepts. He described the challenge in supporting M-PSTs as they worked with authentic real-world situations that were messier, less precise, and required more decision-making before reporting a solution. His goal was to help M-PSTs "make assumptions, approximate certain things, … [and] to use mathematical analysis to derive some conclusion [and] analyze the situation." For example, this instructor described using the U.S. food stamp program (supplemental nutrition assistance program) and its need to determine "a way to meet this minimum requirement for nutrients with minimum cost" based on dietary recommendations, ranges of food costs, Consumer Price Index, and other considerations."

M-PSTs also had opportunities to consider ways of using algebra concepts to support secondary mathematics students in making sense of their world. For example, the GLU secondary mathematics methods instructor reported that "we had a teacher assistant placed in City B, and they were doing a walk-a-thon … And she's like, 'oh, maybe we can keep track of how much money they're making per mile'." An M-PST at GLU commented,

> I think there's a large push in a lot of our classes. I mean, you have secondary mathematics methods … where there's a desire to make students have those connections outside of the math classroom where it's not just confined where I tell you this is how it works, and you go outside and forget it.

The insights of M-PSTs and learning opportunities provided for M-PSTs described here foreground the use of algebra as a tool to make sense of real-world contexts and problems.

Real-world connections used to motivate interest in algebra

In both mathematics and mathematics education courses, instructors used examples of how algebra is used in the real-world or in other subjects to motivate students to learn algebra. Secondary mathematics methods instructors and M-PSTs particularly discussed using meaningful contexts to motivate their students. The secondary mathematics methods II instructor at SRU explained that the M-PSTs tried to find contexts that would motivate their students: “they try to connect mathematics to sports, to buying things, the kind of simplistic ideas that they think middle school kids will look at.” A GLU student teaching seminar instructor described his M-PSTs “trying to justify school algebra to their students, in terms of what they’ll need when they go to college.” An M-PST at GLU said,

> There’s a push to make those connections to non-mathematical themes where it puts things into context for students in ways that they can see, algebra can be used in more places than just in the math classroom … context with different sports, different arts, history where they can go to different classes or different parts of their life to see these mathematical connections.

Another GLU M-PST responded, “I get the question all the time of ‘When are we ever going to use this?’ … you might not be able to find an exact answer that they would care about, but at least you can take it, and you can apply it more than just in the math classroom and I think that that’s something.” These M-PSTs considered using a context as a buy-in and engagement tool for students to learn algebra. In the interviews, the instructors and M-PSTs described the benefits of connecting algebra to non-mathematical contexts, especially real-world situations, to enhance learning experiences for M-PSTs and their future students.

DISCUSSION AND CONCLUSION

Algebra is a foundation of advanced mathematics and serves as a crucial prerequisite for post-secondary education, making it essential to ensure access to algebra for all students (Rech & Harrington, 2000; Teuscher et al., 2008). To open access, teachers need a profound understanding of algebra and the interconnected nature of mathematical ideas (AMTE, 2017; NCTM, 2000). Building on prior studies and policy recommendations (CBMS, 2012; Kieran, 2007; NCTM, 2000; Usiskin, 2015), we investigated how M-PSTs and instructors of required courses in secondary preparation programs conceptualized algebraic connections. We aimed to gain understanding of these connections and their implications for teaching and learning algebra while honoring voices from M-PSTs and the instructors who teach them.

Our findings highlight the multifaceted nature of algebra connections. Instructors and M-PSTs described algebra as both a foundation for other mathematics and a complex web of concepts, processes, and structures. These depictions emphasize the interconnectedness and interdependence of algebraic ideas, highlighting the need to view algebra holistically rather than as a collection of isolated ideas. Viewing algebra as a cohesive system (McCrory et al., 2012) can enhance instruction and support M-PSTs in developing deeper algebraic insights.

The significance of other connections between school and college-level algebra also emerged from our findings (CBMS, 2012; Foley, 1997). Instructors and M-PSTs described drawing on high school algebra in ways that support M-PSTs’ future experiences, particularly in college-level mathematics courses. Instructors and M-PSTs acknowledged the presence of algebraic representations and structures across mathematical domains (CBMS, 2012), pointing out relationships to algebraic objects and concepts in other mathematical fields. They also noticed that algebraic connections could offer insights into real-world situations (Kirshner, 2001; NCTM 2009), enabling M-PSTs to consider real-world connections to illuminate algebraic concepts and using algebra to solve problems in authentic contexts.

Explicit and Implicit Attention to Algebraic Connections

These five secondary mathematics teacher education programs included course objectives that varied in emphasis on algebraic connections. Only mathematics education and mathematics-for-teachers courses at participating research institutions (i.e., Midwestern Research and Southeastern Research) included algebraic connections as explicit objectives in their courses; only mathematics courses at Master’s-degree-offering institutions (i.e., Great Lakes, Midwestern Urban) included these connections as explicit objectives.

Overall, 11 of 48 courses listed one or more algebraic connections as course objectives; other math fields connections were listed as a course objective in seven of 48 courses, followed by within algebra connections and school and college-level connections (both four times), and non-math fields connections (three times). All courses here are required for M-PSTs; hence, it may be productive for mathematics, mathematics education, and mathematics-for-teachers course instructors to discuss how to ensure a variety of experiences supporting building algebraic connections are provided across their programs. Our study can contribute to conversations among teacher educators and administrators who aim to improve what is emphasized in required mathematics, mathematics-for-teachers, and mathematics education courses that are designed specifically for M-PSTs.

Algebra Connections Reported by M-PSTs and Instructors

We observed patterns in instructors’ and M-PSTs’ conceptualizations of algebraic connections. For example, several instructors (e.g., abstract algebra at Great Lakes and Midwestern Urban, algebra in the curriculum at Midwestern Research, and linear algebra at Midwestern Urban) included connections within algebra course objectives or big ideas of the course; however, even at those universities, M-PSTs shared fewer examples of within algebra connections than the other connection types. The outcome revealed that M-PSTs had limited recollection of instances related to within algebra connections, despite instructors’ reports of many connections; this inconsistency highlights the need for future research to delve into the underlying reasons or solutions. In addition, both instructors and M-PSTs described experiences related to making interdisciplinary connections and making real-life connections as described by NCTM (2000; 2009) and in *CCSSM* (NGA & CCSSO, 2010). At each university, M-PSTs reported non-math fields connections most often compared to other types of algebraic connections; this might suggest that these Connections were memorable to M-PSTs. They described connections between history and algebra (Great Lakes), explained an experience to gather scientific data to learn about logarithms (Southeastern Research), and described connections between mathematics and English (Western Urban). Future study may include an investigation of the factors that influenced M-PSTs' emphasis on this type of connection.

While school and college-level connections were reported least often by instructors compared to other algebraic connections, school and college-level was the second most often reported by focus group M-PSTs. Although M-PSTs mentioned this type of experience slightly more often than within algebra and other math fields connections in total, it was much less often mentioned than non-math fields connections. Additionally, M-PSTs who took linking courses either did not mention school and college-level connections or mentioned them only a few examples. The results were consistent with findings from the national survey that the majority of secondary mathematics teacher education programs do not meet the recommendations from CBMS (2012) regarding school and college-level connections; more such connection-making experiences for M-PSTs are needed.

Implications for Research and Teacher Education

Overall, course instructors made efforts to incorporate attention to algebraic connections in courses for M-PSTs. The findings suggest, however, the need for more purposeful work in this area, likely requiring coordination between instructors and departments serving the M-PSTs in the program to more fully address all types of algebraic connections in ways that are appropriate within their context. In particular, connections between school and college-level algebra were underrepresented as in the PTA survey (Newton et al., 2024); perhaps course development collaborations are needed to meet this recommendation. Future studies might delve deeper into the specific instructional strategies that prove effective in fostering M-PSTs' robust understanding connections between school and college-level algebra. Additionally, investigating the long-term impact of such opportunities on these M-PSTs' classroom practices and their students' learning would provide valuable insights into the overall effectiveness of teacher education programs in this critical domain. Although only five universities were included in this investigation, the findings from focus groups with M-PSTs in each program and 48 instructor interviews provide ample information about and instances of attention to algebraic connections to contribute to our understanding of how M-PSTs' understandings of these connections are being conceptualized, promoted, and supported throughout their course work.

Our study contributes to the understanding of how instructors of secondary preparation programs and M-PSTs conceptualize and promote algebraic connections. Specifically, we investigated experiences provided for M-PSTs to learn about algebraic connections in secondary mathematics teacher education programs. We tried to enhance the reliability of our data analysis by systematically examining three distinct sources of information: implemented course materials, interviews with instructors, and interviews with M-PSTs. Our findings extend extant research and implications for mathematics teacher preparation programs in three ways:

(a) highlight varying perspectives of M-PSTs and instructors related to encounters with algebraic connections,

(b) report on algebraic connection types less often emphasized by participants,

(c) present the types of algebraic connections emphasized as the main objectives of courses in five secondary mathematics teacher education programs, and

(d) nine emergent themes that emerged from the five teacher education programs.

These findings are critical to continue to understand and improve the preparation of teachers to comprehend algebraic connections in diverse programs across the United States. The findings of this study also carry important implications for curriculum development, instructional practices, and the fundamental concepts of algebra learning and teaching in teacher education programs.

Author contributions: All authors contributed equally to the study and agreed with the results and conclusions.

Funding: This study was initiated through *Preparing to Teach Algebra (PTA)* and extended with *Exploring Opportunities to Learn about Algebraic Modeling, Connections, and Technology (MCT). PTA* is a collaborative project at Michigan State University and Purdue University funded through NSF's REESE program (MSU 1109256, Sharon L. Senk, PI; Purdue 1109239, Jill Newton, PI, Yukiko Maeda, Co-PI). Any opinions, findings, conclusions or recommendations expressed in this article are those of the authors and do not necessarily reflect the views of the National Science Foundation. *MCT* is a collaborative project funded by the Spencer Foundation (Jill Newton, PI). Any opinions, findings, conclusions or recommendations expressed in this article are those of the authors and do not necessarily reflect the views of the Spencer Foundation.

Ethical statement: The authors stated that the study was not a study of human subject; rather, it was a study of teacher education programs. Informed consents were obtained from the participants.

Declaration of interest: No conflict of interest is declared by the authors.

Data sharing statement: Data supporting the findings and conclusions are available upon request from the corresponding author.

REFERENCES

AMTE. (2017). Standards for preparing teachers of mathematics. *Association of Mathematics Teacher Educators*. http://amte.net/standards

Artzt, A. F., Sultan, A., Curcio, F. R., & Gurl, T. (2012). A capstone mathematics course for prospective secondary mathematics teachers. *Journal of Mathematics Teacher Education, 15*(3), 251-262. https://doi.org/10.1007/s10857-011-9189-5

Bremigan, E. G., Bremigan, R. J., & Lorch, J. D. (2011). *Mathematics for secondary school teachers*. MAA Press. https://doi.org/10.1090/text/021

Carraher, D. W., & Schliemann, A. D. (2007). Early algebra and algebraic reasoning. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (vol. 2, pp. 669-705). Information Age.

CBMS. (2012). The mathematical education of teachers II. *Conference Board of the Mathematical Sciences*. https://www.cbmsweb.org/archive/MET2/met2.pdf

Cuoco, A., & Rotman, J. J. (2013). *Learning modern algebra: From early attempts to prove Fermat's last theorem*. Mathematical Association of America.

Foley, G. D. (1997). The role of algebraic structure in the mathematics curriculum of grades 11-14. In *The nature and role of algebra in the K-14 curriculum* (pp. 87-88). National Academy.

García-García, J., & Dolores-Flores, C. (2021). Pre-university students' mathematical connections when sketching the graph of derivative and antiderivative functions. *Mathematics Education Research Journal, 33*, 1-22. https://doi.org/10.1007/s13394-019-00286-x

Glaser, B. G. (1965). The constant comparative method of qualitative analysis. *Social Problems, 12*(4), 436-445. https://doi.org/10.2307/798843

Gravemeijer, K. (2002). *Symbolizing, modeling, and tool use in mathematics education*. Kluwer. https://doi.org/10.1007/978-94-017-3194-2

Haciomeroglu, E. S. (2007). *Calculus students' understanding of derivative graphs problems of representations in calculus* [Unpublished dissertation]. Florida State University.

Henry, G. T., Fortner, C. K., & Bastian, K. C. (2012). The effects of experience and attrition for novice high-school science and mathematics teachers. *Science, 335*(6072), 1118-1121. https://doi.org/10.1126/science.1215343

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 707-762). Information Age.

Kieran, C. (2013). The false dichotomy in mathematics education between conceptual understanding and procedural skills: An example from algebra. In K. R. Leatham (Ed.), *Vital directions for mathematics education research* (pp. 153-171). Springer. https://doi.org/10.1007/978-1-4614-6977-3_7

Kilpatrick, J, & Izsák, A. (2008). A history of algebra in the school curriculum. In C. E. Greenes, & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics* (pp. 1-18). National Council of Teachers of Mathematics.

Kirshner, D. (2001). The structural algebra option revisited. In R. Sutherland, T. Rojano, A. Bell, & R. C. Lins (Eds.), *Perspectives on school algebra* (pp. 83-98). Kluwer. https://doi.org/10.1007/0-306-47223-6_5

Landis, J., & Koch, G. (1977). The measurement of observer agreement for categorical data. *Biometrics, 33*(1), 159-174. https://doi.org/10.2307/2529310

Lehrer, R., & Schauble, L. (2000). Modeling in mathematics and science. In R. Glaser (Ed.), *Advances in instructional psychology: Vol. 5. Educational design and cognitive science.* Lawrence Erlbaum.

McCrory, R., Floden, R., Ferrini-Mundy, J., Reckase, M. D., & Senk, S. L. (2012). Knowledge of algebra for teaching: A framework of knowledge and practices*. Journal for Research in Mathematics Education, 43*(5), 584-615. https://doi.org/10.5951/jresematheduc.43.5.0584

Moore-Harris, B. (1998). Improving K-14 algebra instruction: A discussion of teachers' responsibilities and students' opportunities. In *The nature and role of algebra in the K-14 curriculum* (pp. 45-47). National Academy.

Moses, R. P., & Cobb, C. E. (2001). *Radical equations*. Beacon.

National Research Council. (2010). *Preparing teachers: Building evidence for sound policy*. National Academy.

NCTM. (2000). *Principles and standards for school mathematics*. National Council of Teachers of Mathematics.

NCTM. (2009). *Focus in high school mathematics: Reasoning and sense making*. National Council of Teachers of Mathematics.

NCTM. (2012). NCATE/NCTM program standards: Programs for initial preparation of mathematics teachers. *National Council of Teachers of Mathematics*. http://www.nctm.org/uploadedFiles/Math_Standards/NCTM%20NCATE%20Standards%20Revision%20and%20Math%20Content%20-%20Secondary.pdf

Newton, J., Maeda, Y., Alexander, V., & Senk, S. L. (2014). How well are secondary mathematics teacher education programs aligned with the recommendations made in MET II? *Notices of the American Mathematical Society, 61*(3), 292-295. https://doi.org/10.1090/noti1089

NGA & CCSSO. (2010). Common core state standards for mathematics. *National Governors Association Center for Best Practices & Council of Chief State School Officers*. http://www.corestandards.org/Math/

Rech, J. F., & Harrington, J. (2000). Algebra as a gatekeeper: A descriptive study at an urban university. *Journal of African American Studies, 4(*4), 63-71. https://doi.org/10.1007/s12111-000-1022-7

Sultan, A., & Artzt, A. F. (2010). *The mathematics that every secondary school math teacher needs to know*. Routledge. https://doi.org/10.4324/9780203857533

Teuscher, D., Dingman, S. W., Nevels, N. N., & Reys, B. J. (2008). Curriculum standards, report of state policies. *Journal of Mathematics Education Leadership, 10*(2), 49-55.

Usiskin, Z. (2015). What does it mean to understand some mathematics? In *Proceedings of the 12th International Congress on Mathematical Education* (pp. 821-841). Springer. https://doi.org/10.1007/978-3-319-17187-6_46

Usiskin, Z., Peressini, A. L., Marchisotto, E., & Stanley, D. (2002). *Mathematics for high school teachers: An advanced perspective.* Pearson.

Weiss, M. (2021). *Secondary mathematics for mathematicians and educators: A view from above*. Routledge. https://doi.org/10.4324/9781315100968